Biomechanics of the musculoskeletal system

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What do we want to obtain with biomechanical studies?

- Calculate the stress in the tissues
- Evaluate the implant effect on joint kinematics
- Anticipate the tissue differentiation due to the mechanical loading

We have previously determined the “boundary conditions” (joint forces, kinematics) acting on our system of interest (forearm, femur, shoulder, …). We can use this information to calculate the corresponding stress or strain inside the different tissues of the joint. We are then at one level deeper in our biomechanical description and we can use this new knowledge to evaluate the biomechanical impact of an implant on the surrounding tissues. If an evolution law of the tissue mechanical properties is developed in function of a mechanical stimulus, we would then also be able to anticipate the tissue differentiation with respect to this mechanical stimulus.

Biomechanics at the tissue level

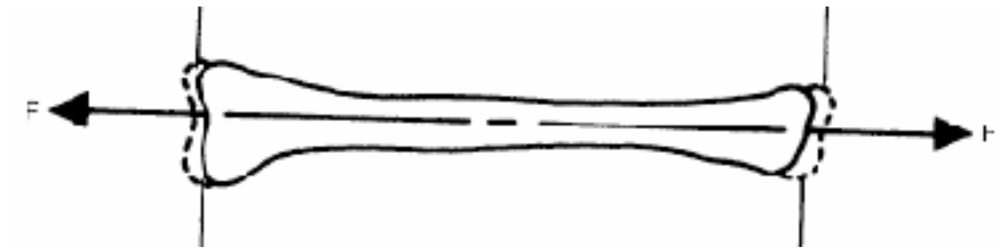
i) Continuum mechanics (conservation laws)

ii) Constitutive laws (linear, non-linear)

iii) Tissue characterisation

Like in any field of physics, in the new description of mechanics at the continuum level, conservation laws have to be established.

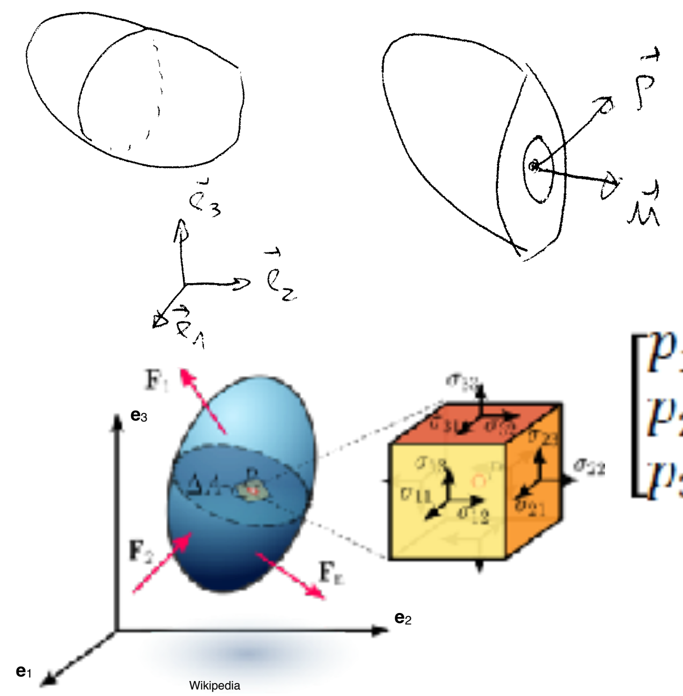
Biological tissues are stressed and deformed when a force or moment is applied



Newton equations of motion $m\vec{a} = \sum \vec{F}^{ext}$ $I\dot{\omega} = \sum \vec{M}_o^{ext}$

So far, we have considered the tissues forming the musculoskeletal system as rigid. Forces and moments are applied to these structures and if not in a static situation, their effects are to put them into linear/angular motions (studied through kinematics). In fact, these structures are deformable. Tissues such as ligaments, tendons, cartilage, meniscus, skin are highly deformable, while bone is less deformable. These deformations play an important role in the biomechanical aspects and we must therefore consider biological tissues as deformable. The force/moment will then also induce deformations in the tissues, so the concepts of the stress and strain have to be introduced.

The mathematical “*nature*” of a stress and a strain is a second order tensor



$$\boldsymbol{p} = \boldsymbol{\sigma n}$$

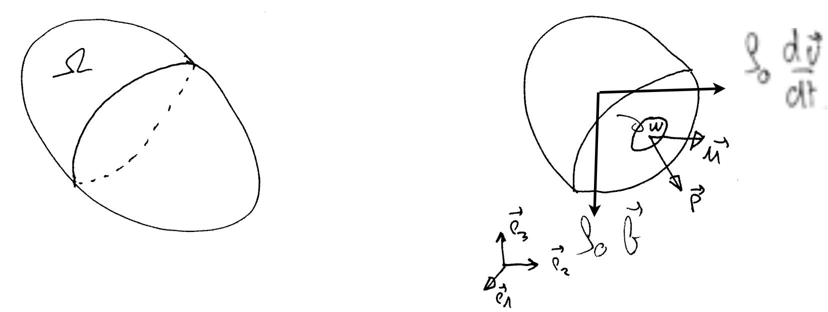
$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

σ_{ij} : stress tensor components

Imagine that a body is virtually cut in two pieces. The second part exerts a force **p** on the surface ∂w of the first part at a certain material point k. The surface ∂w is represented by its normal vector **n**. The force **p** can be understood as the cohesion force which keeps the body in one part. We call the cohesion force **p**: the stress vector. We can repeat the cutting at the material point k with a different orientation, we will then obtain another surface ∂w’. The corresponding stress vector **p**’ will then be different from **p**. The stress vector across any imaginary surface depends then on the orientation of that surface. Cauchy was the first one to propose that the stress vector **p** across a surface will always be a linear function of the surface's normal vector **n**. So **p=σn**.

We can call the “object” **σ** stress and its mathematical “nature” is then revealed as a second order tensor (by definition a second order tensor is a geometric object that linearly transforms a vector (also called first order tensor) into a vector. The components of a second order tensor is expressed in a 3x3 matrix which values depend on the chosen coordinate system. By similitude, the strain is also a second order tensor.

Conservation of the linear momentum



$$\frac{d}{dt}\int_{\omega} \rho_0 \vec{v}\, dV = \int_{\partial\omega} \vec{p}\, dA + \int_{\omega} \rho_0 \vec{b}\, dV \qquad \forall\ \omega \in \Omega$$

There are two kinds of forces to be considered in the conservation of the linear momentum: the forces of volume and the forces of contact.
The left hand-side of the equation represents the force of inertia by unit of reference volume. It is a force of volume. **b** is another force of volume defined by unit of reference volume. Typically, **b** is the gravity force. **p** is a contact force that we just have called stress vector in the previous slide. As mentioned before, this stress vector is assumed to depend only linearly on the unit normal of the surface: **p** = **p**(**x**,t,**n**(x)). The Cauchy theorem allows us to write: **p**(**x**,t,**n**(**x**)) = **σ**(**x**,t) **n**(**x**) where **σ** has just been called the stress tensor.

Conservation of the linear momentum (surface to volume integration)

$$\frac{d}{dt}\int_{\omega} \rho_0 \vec{v}\, dV = \int_{\partial\omega} \sigma \vec{n}\, dA + \int_{\omega} \rho_0 \vec{b}\, dV \qquad \forall \omega \in \Omega$$

Divergence theorem: $\int_{\partial\omega} \sigma \vec{n}\, dA = \int_{\omega} \operatorname{div} \sigma\, dV$

$$\Rightarrow \frac{d}{dt}\int_{\omega} \rho_0 \vec{v}\, dV = \int_{\omega} \operatorname{div} \sigma\, dV + \int_{\omega} \rho_0 \vec{b}\, dV \qquad \forall \omega \in \Omega$$

We replace "**p**(**x**,t,**n**(**x**))" by "**σ**(**x**,t) **n**(x)" and with the help of the divergence theorem, we can transform the surface integral into a volume integral.

Conservation of the linear momentum (localisation)

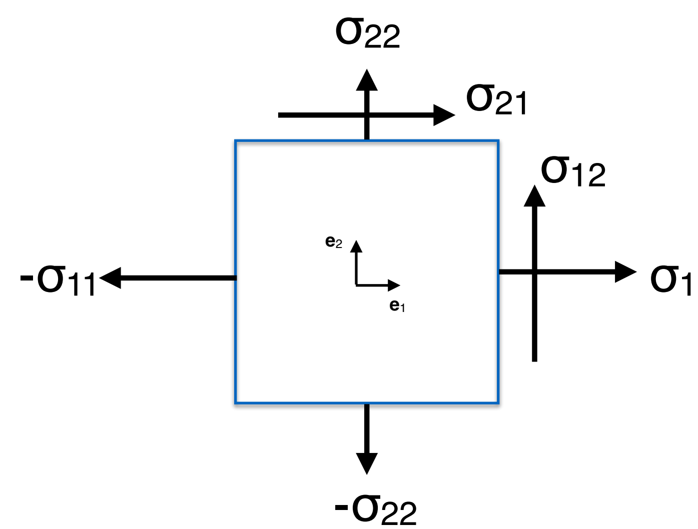
$$\Rightarrow \frac{d}{dt}\int_\omega \rho_0 \vec{v}\, dV = \int_\omega \operatorname{div}\sigma\, dV + \int_\omega \rho_0 \vec{b}\, dV \quad \forall \omega \in \Omega$$

$$\boxed{\rho_0 \frac{d\vec{v}}{dt} = \operatorname{div}\sigma + \rho_0 \vec{b}}$$

The principle of localisation in a continuum medium states that if the integral of a function continue on Ω vanishes in any integration sub-domain ω included in Ω, so this function (the integrant) is null in all the Ω domain.

The conservation of the linear momentum (which is indeed a rewriting of the second Newton law m**a** = $\sum$**F**^{ext}, but for a continuum medium) in continuum mechanics imposes then to define an ad hoc relationship between the stress and the strain (called constitutive law). This constitutive law is specific to the material studied.

Conservation of the angular momentum: graphical interpretation

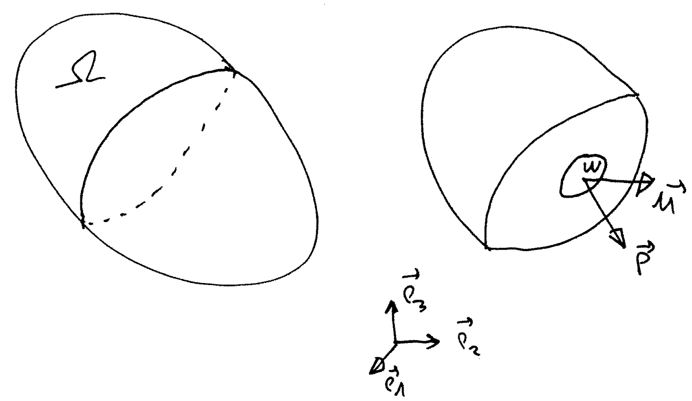


The satisfaction of the angular conservation momentum imposes:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

Let's imagine that we have a body at equilibrium (linear and angular momenta are trivial). If we virtually isolate an infinitesimal part of this body, we can evaluate the stress state situation in this part and make the intuitive conclusion that $\sigma_{12} = \sigma_{21}$. In other word, this observation suggests that the stress (and correspondingly strain) tensor should be symmetrical.

Conservation of the angular momentum



$$\frac{d}{dt}\int_{\omega} \vec{x} \wedge \rho_0 \vec{v}\, dV = \int_{\partial\omega} \vec{x} \wedge \vec{p}\, dA + \int_{\omega} \vec{x} \wedge \rho_0 \vec{b}\, dV \quad \forall \omega \in \Omega$$

To get a formal demonstration of the symmetry of the stress tensor imposed by the conservation of the angular momentum, we extensively write the equation of this conservation law in the framework of continuum mechanics.

Conservation of the angular momentum (math development)

$$\frac{d}{dt}\int_{\omega} \rho_0 \vec{x} \wedge \vec{v}\, dV = \int_{\partial\omega} \vec{x} \wedge (\sigma \vec{n})\, dA + \int_{\omega} \rho_0 \vec{x} \wedge \vec{b}\, dV \quad \forall \omega \in \Omega$$

$$\Rightarrow i) \quad \frac{d}{dt}\int_{\omega} \rho_0 \vec{x} \wedge \vec{v}\, dV = \rho_0 \int_{\omega} \frac{d\vec{x}}{dt} \wedge \vec{v}\, dV + \rho_0 \int_{\omega} \vec{x} \wedge \frac{d\vec{v}}{dt}\, dV$$

$$(\rightarrow \vec{v} \wedge \vec{v} = 0)$$

$$\text{"}=\text{"} \rho_0 \int_{\omega} \varepsilon_{ijk}\, x_j\, v_k\, dV$$

$$\varepsilon_{ijk} = \begin{cases} 1 \text{ if } (i,j,k) \text{ is an even permutation} \\ \quad (\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1) \\ -1 \text{ if } (i,j,k) \text{ is an odd permutation} \\ \quad (\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1) \\ 0 \text{ if any } (i,j,k) \text{ are equal} \end{cases}$$

$$\varepsilon_{112} = \varepsilon_{333} = \ldots = 0$$

We develop each part of the equation in components. To express the vector product in components, we use the permutation symbol ε_{ijk}.

Conservation of the angular momentum (math development)

$$\frac{d}{dt}\int_{\omega} \rho_0 \, \vec{x} \wedge \vec{v} \, dV = \int_{\partial\omega} \vec{x} \wedge (\sigma \vec{n}) \, dA + \int_{\omega} \rho_0 \, \vec{x} \wedge \vec{b} \, dV \quad \forall \omega \in \Omega$$

$$(i) \int_{\partial\omega} \varepsilon_{ijk} \, x_j \, \sigma_{km} \, n_m \, dA$$

$$= \int_{\omega} \varepsilon_{ijk} \, (x_j \sigma_{km})_{,m} \, dV = \int_{\omega} \varepsilon_{ijk} (\sigma_{kj} + x_j \sigma_{km,m}) \, dV$$

By using the divergence theorem, we transform the vector product expressed with a surface integral in a volume integral.

Conservation of the angular momentum (math development)

$$\frac{d}{dt}\int_{\omega} \rho_0\, \vec{x} \wedge \vec{v}\, dV = \int_{\partial\omega} \vec{x} \wedge (\sigma\, \vec{n})\, dA + \int_{\omega} \rho_0\, \vec{x} \wedge \vec{b}\, dV \quad \forall \omega \in \Omega$$

$$\text{(iii)} \int_{\omega} \rho_0\, \vec{x} \wedge \vec{b}\, dV \text{ "=" } \int_{\omega} \rho_0\, \varepsilon_{ijk}\, x_j\, b_k\, dV$$

$$\Rightarrow \int_{\omega} \left\{ \varepsilon_{ijk}\, x_j \left(\rho_0 \dot{v}_k - \sigma_{km,m} - \rho_0 b_k \right) - \varepsilon_{ijk}\, \sigma_{kj} \right\} dV = 0$$

We can now express the conservation of the angular momentum in components form and we see that we have basically two parts.

Conservation of the angular momentum (math development)

$$\boxed{\int_W \left\{ \varepsilon_{ijk}\, x_j \left(\rho_0 \dot{v}_k - T_{km,m} - \rho_0 b_k\right) - \left(\varepsilon_{ijk} T_{kj}\right) \right\} dV = 0}$$

$$\rho_0 \dot{v}_k - T_{km,m} - \rho_0 b_k = 0 \quad \text{(conservation of the linear momentum)}$$

$$\Rightarrow \boxed{\varepsilon_{ijk} T_{kj} = 0}$$

$$i = 1 \quad \underbrace{\varepsilon_{123}}_{1} T_{32} + \underbrace{\varepsilon_{132}}_{-1} T_{23} = 0$$

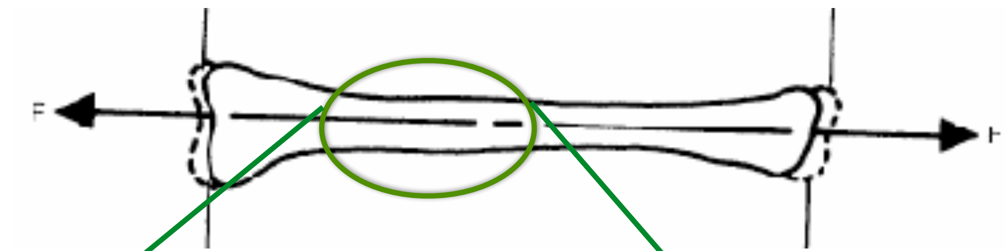
$$T_{32} - T_{23} = 0 \qquad \boxed{T_{32} = T_{23}}$$

$$i = 2 \ldots \quad T_{31} = T_{13}$$

$$i = 3 \ldots \quad T_{12} = T_{21}$$

The first part of the equation is nothing else than the equation for the conservation of linear momentum which is trivial. To satisfy the conservation of the angular momentum, the term into bracket of the second part must be equal to 0. This is generally obtained if the stress tensor is symmetric. As for the conservation of the linear momentum, the principle of localisation allows us to consider only the integrant.

Biological tissues are stressed and deformed when a force or moment is applied



Newton equations of motion $m\vec{a} = \sum \vec{F}^{ext}$ $I\dot{\omega} = \sum \vec{M}_o^{ext}$

Cauchy momentum equation $\rho_0 \frac{\partial v}{\partial t} = \nabla \cdot \sigma + \rho b$

$\sigma = \sigma^T$

Unlike the equations of motion proposed by Newton which considers only external forces/moments to the defined system, in deformable materials (which are considered as a continuum here), internal forces/moments (represented by stress) are part of the conservation of the linear and angular momenta. These conservation laws are called the Cauchy momentum equations.

Biomechanics at the tissue level

i) Continuum mechanics (conservation laws)

ii) Constitutive laws (linear, non-linear)

iii) Tissue characterisation

One important aspect of biomechanics is then to characterise tissues through constitutive laws

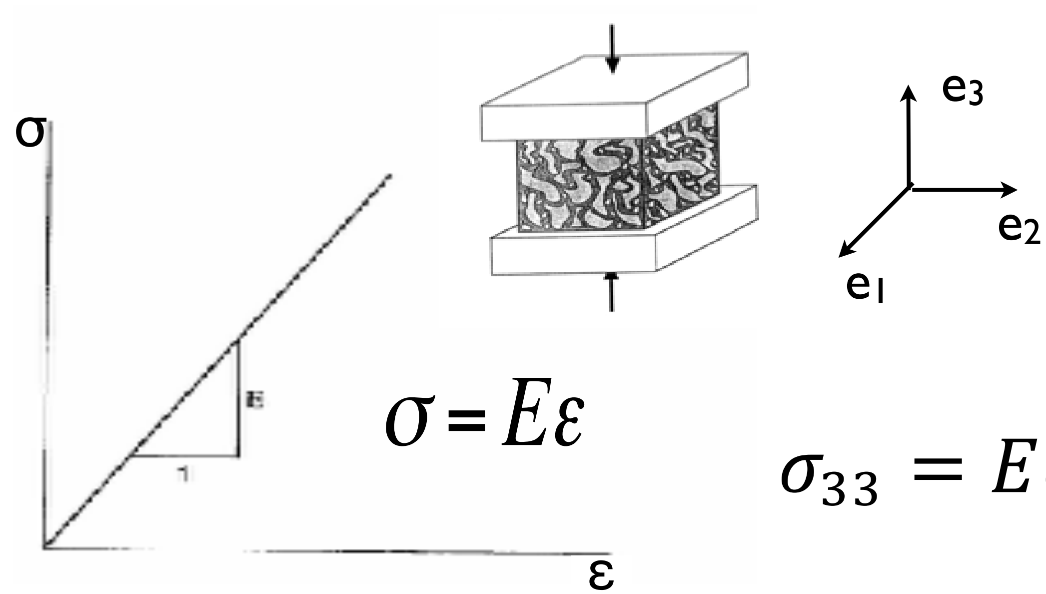
$$\rho_0 \frac{d\boldsymbol{v}}{dt} = div\,\boldsymbol{\sigma} + \rho \boldsymbol{b}$$

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}, \varepsilon_p, \dots)$$

Elasticity -> $\sigma = \sigma(\varepsilon)$ → Linear / Non-linear

Conservation laws (linear or angular momenta, mass conservation, energy and second law of thermodynamics) are general and in order to characterise the mechanical behaviour of particular materials, constitutive laws have to be postulated. The postulation of a constitutive law is then an ad hoc way to describe a particular (mechanical) behaviour which cannot be taken into consideration by the general conservation laws.

Hooke's Law in 1D



$$\sigma_{33} = E\epsilon_{33}$$

E: Young's modulus

The stress/strain curve presented so far, corresponds to experimental test performed in a one dimension. This is a shortcut as the tested samples have obviously a 3D structure. To be correct, the relation between the stress and the strain should then also indicate with respect to which axis the force was applied. In the example of this slide, the indication is given by mentioning the (second) indice 3 for the stress and the strain referring then that the force was applied in the direction of the axis $\mathbf{e}_3$. As a stress is by definition a force divided by a surface, we need to precise on which surface we consider that the force is applied. The first indices describes the direction of the normal to the plane on which the force acts. Thus, σ_{12} indicates a stress component acting in 2-direction on1-plane. When both the indices are same, it means the stress component is along the normal to the plane on which it acts. It is called the normal stress component. Thus, σ_{11}, σ_{22} and σ_{33} are the normal stress components. When the two indices are different, it means the direction of the component is within the plane. Such a component is called the shear stress component. Linear relationship between the stress and strain is called Hooke's law.

Hooke law in 3D (symmetries of the stiffness tensor **C**)

General linear relationship between the stress and the strain (**81** parameters)

$$\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl}$$

Symmetry of the stress and corresponding strain tensors (**36** parameters)

$$C_{ijkl} \rightarrow C_{\alpha\beta}$$

When formalising the linear relationship between the stress (second order tensor) and strain (second order tensor), we obtain that the elastic constants ("proportional factors") are represented by a fourth order tensor (3x3x3x3) meaning that in the general form, it involves 81 parameters. The stiffness tensor is a fourth order tensor.

The symmetry of the stress tensor ($\sigma_{ij} = \sigma_{ji}$) and the generalised Hooke's laws ($\sigma_{ij} = C_{ijkl}\ \varepsilon_{kl}$) implies that $C_{ijkl} = C_{jikl}$. Similarly, the symmetry of the strain tensor implies that $C_{ijkl} = C_{ijlk}$. These symmetries are called the minor symmetries of the stiffness tensor (**C**). This reduces the number of elastic constants from 81 to 36.

Matrix notation of Hooke's law (Voigt notation)

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \; ; \; [\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

$$[\sigma] = [C][\epsilon]$$

$$[C] = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix} := \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix}$$

As the stress and strain tensors are symmetric, on the 9 components composing the 3x3 matrix form of these tensors, only 6 components are independent. It has then be proposed to represent these second order tensors, in a vector form of 6 components. To be coherent with the definition of the Hooke's law, a factor 2 must be used for the off-diagonal component of the strain in the transformation from two indices notation to a one indice notation. Accordingly, the stiffness tensor is represented in a 6x6 matrix.

Hooke law in 3D (symmetries of the stiffness tensor C)

Stress derived from a strain energy function U (**21** parameters)

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} \Longrightarrow C_{ijkl} = \frac{\partial^2 U}{\partial \epsilon_{ij} \partial \epsilon_{kl}}$$

$$C_{\alpha\beta} \text{ -> } C_{\beta\alpha}$$

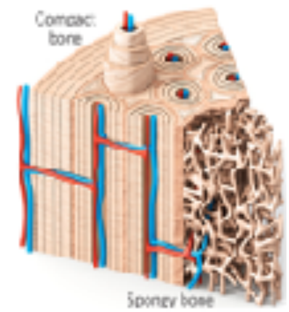
$$[C] := \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{55} & C_{66} \end{bmatrix} := \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

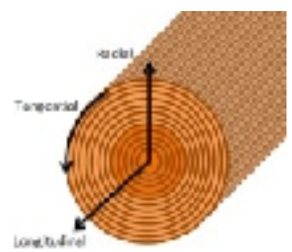
$$\sigma_i = C_{ij} \varepsilon_j$$

Finally, the stress-strain relation can be derived from a strain energy density functional U. The arbitrariness of the order of differentiation implies that $C_{ijkl} = C_{klij}$. These are called the major symmetries of the stiffness tensor. This reduces the number of elastic constants from 36 to 21. The major and minor symmetries indicate that the stiffness tensor has at maximum only 21 independent components.

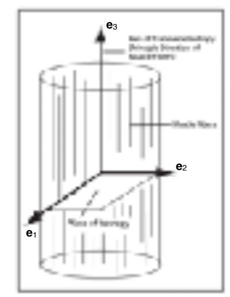
Hooke law in 3D (material symmetry)

Orthotropic
(3 plans)

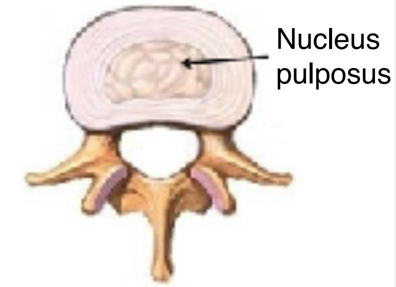


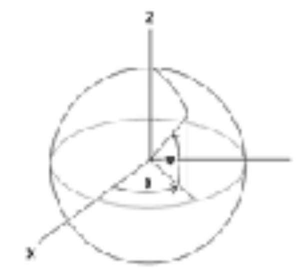


Transversely isotropic
(1 direction perpendicular to a plan)



Isotropic
(no symmetry)





Thanks to their structure, musculoskeletal tissues may present specific symmetries with respect to their mechanical behaviour. In particular for the linear elastic situation, these material symmetries (called anisotropy) or absence of symmetries (called isotropy) will affect the number of independent parameters in the stiffness tensor. In other words, the mechanical properties of a material also obviously depend on its intrinsic symmetries.

Hooke law in 3D (material symmetry)

Isotropic material -> isotropic stiffness tensor (**2** parameters):

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

δ: Kronecker symbol

λ and μ are 2 scalars called <u>Lamé constants</u>

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0 & 0 & 0\\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0\\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0\\ 0 & 0 & 0 & \mu & 0 & 0\\ 0 & 0 & 0 & 0 & \mu & 0\\ 0 & 0 & 0 & 0 & 0 & \mu\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_3\\\varepsilon_4\\\epsilon_5\\\varepsilon_6\end{bmatrix}$$

The class of materials which mechanical properties do not depend on the direction is said to be isotropic. This concept of isotropy is used constantly in (bio)mechanics as a simplifying assumption. If an elastic solid is considered as isotropic, then its stiffness tensor must be isotropic. The values of the elastic parameters do not vary under any applied orthogonal transformation. An isotropic tensor of rank 4 can then reduce to the proposed form written in the slide. Only two independent parameters are necessary to fully characterise the linear mechanical behaviour of an isotropic material.

Tensorial formulation for linear elastic isotropic material

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$$

$$\mathrm{tr}\boldsymbol{\varepsilon} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$\mathbf{I}$: tensor identity

link with "usual" E (Young's modulus)

$\sigma_{33} = E\varepsilon_{33}$?

We can obtain a direct tensor notation for an isotropic elastic material with the Lamé constants (λ and μ). Different materials considered as linear elastic isotropic will then have different values for the constants λ and μ. However, in linear elasticity we may use other (well-known) constants such as E the Young's modulus (also called elastic modulus), ν the Poisson ratio, or even k the bulk modulus (and G the shear modulus which is equivalent to μ).

Hooke law in 3D (material symmetry -> **isotropy**)

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \epsilon_5\\ \varepsilon_6\end{bmatrix} = 1/E\begin{bmatrix}1 & -\upsilon & -\upsilon & 0 & 0 & 0\\ -\upsilon & 1 & -\upsilon & 0 & 0 & 0\\ -\upsilon & -\upsilon & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 2(1+\upsilon) & 0 & 0\\ 0 & 0 & 0 & 0 & 2(1+\upsilon) & 0\\ 0 & 0 & 0 & 0 & 0 & 2(1+\upsilon)\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix}$$

Tensorial formulation -> $\boldsymbol{\varepsilon} = {}^{1}/_{E}\,(\boldsymbol{\sigma} - \upsilon[tr(\sigma)\boldsymbol{I} - \boldsymbol{\sigma}])$

Voigt notation (3) -> $\varepsilon_3 = {}^{1}/_{E}\,[\sigma_3 - \upsilon(\sigma_1 + \sigma_2)]$

Index notation (33) -> $\varepsilon_{33} = {}^{1}/_{E}\,[\sigma_{33} - \upsilon(\sigma_{11} + \sigma_{22})]$

The general formulation of Hooke law in 3D using the Young's modulus and the Poisson ratio takes a simpler form in the strain-stress representation (also called compliance form) and the proportional parameters in this representation formed what is called a compliance matrix. A straightforward tensorial formulation can be obtained from which the index notation is derived. We can then see that in an experimental test where the force is acting in the $\mathbf{e}_3$ direction perpendicular to the surface given by the normal co-linear to $\mathbf{e}_3$ and if no lateral constraint is imposed on the sample (-> σ_{11} and σ_{22} are trivial), we obtained the relationship $\sigma_{33}= E\varepsilon_{33}$.

Relation between the different isotropic elastic linear parameters

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix}1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0\\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0\\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{(1-2\upsilon)}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{(1-2\upsilon)}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \epsilon_5\\ \varepsilon_6\end{bmatrix}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

The stress-strain relationship of Hooke's law (also called stiffness form) using E and ν is less elegant. As there are only 2 independent parameters needed to describe the linear elastic behaviour of an isotropic material, relationships should exist between the different sets of two parameters. The relationships between the two Lamé constants (λ, μ) and the set (E, ν) is easily obtained. From a conceptual point of view, the choice of the two independent parameters is completely open and depends on the experimental data available or the convenience of one description over the others.

Relation between the different isotropic elastic linear parameters

	λ	μ	E	ν	k
λ, μ	–	–	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{3\lambda+2\mu}{3}$
λ, ν	–	$\frac{\lambda(1-2\nu)}{2\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\frac{\lambda(1+\nu)}{3\nu}$
λ, k	–	$\frac{3(k-\lambda)}{2}$	$\frac{9k(k-\lambda)}{3k-\lambda}$	$\frac{\lambda}{3k-\lambda}$	–
μ, E	$\frac{(2\mu-E)\mu}{(E-3\mu)}$	–	–	$\frac{(E-2\mu)}{2\mu}$	$\frac{\mu E}{3(3\mu-E)}$
μ, ν	$\frac{2\mu\nu}{1-2\nu}$	–	$2\mu(1+\nu)$	–	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$
μ, k	$\frac{3k-2\mu}{3}$	–	$\frac{9k\mu}{3k+\mu}$	$\frac{3k-2\mu}{6k+2\mu}$	–
E, ν	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	–	–	$\frac{E}{3(1-2\nu)}$
E, k	$\frac{3k(3k-E)}{9k-E}$	$\frac{3kE}{9k-E}$	–	$\frac{(3k-E)}{6k}$	–
ν, k	$\frac{3k\nu}{1+\nu}$	$\frac{3k(1-2\nu)}{2(1+\nu)}$	$3k(1-2\nu)$	–	–

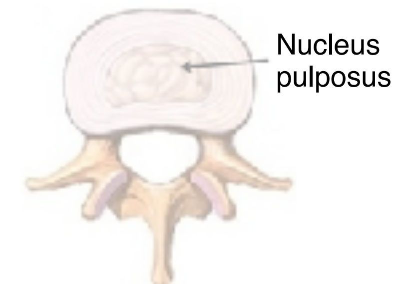
Tissue mechanics, Cowin, 2007

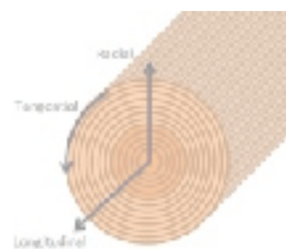
Hooke law in 3D (material symmetry)

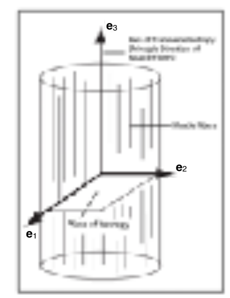
Orthotropy
(3 plans)

Transverse isotropy
(1 direction perpendicular to a plan)

Isotropy
(no symmetry)







A transversely isotropic material is symmetric with respect to a rotation about an axis of symmetry ($\mathbf{e}_3$ in the given example of the tendon here).

Hooke law in 3D (material symmetries -> **transversely isotropic**)

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \epsilon_5 \\ \varepsilon_6 \end{bmatrix} =
\begin{bmatrix}
\frac{1}{E_p} & \frac{\nu_p}{E_p} & \frac{-\nu_t}{E_p} & 0 & 0 & 0 \\
\frac{-\nu_p}{E_p} & \frac{1}{E_p} & \frac{-\nu_t}{E_p} & 0 & 0 & 0 \\
\frac{\nu_t}{E_p} & \frac{\nu_t}{E_p} & \frac{1}{E_t} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G_{tp}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G_{tp}} & 0 \\
0 & 0 & 0 & 0 & 0 & 2\frac{1+\nu_p}{E_p}
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}
$$

E_p, ν_p: Young modulus and Poisson ratio in the plane of isotropy
E_t, ν_t: Young modulus and Poisson ratio in the transverse direction (axis of symmetry)
G_{tp}: shear modulus in the plane of isotropy

As an example for a transversely isotropic material, if $\mathbf{e}_3$ is the axis of symmetry (we call this axis “transverse”), Hooke's law in compliance form can be expressed as shown on this slide. There are 5 independent parameters needed to fully characterise the mechanical behaviour of a transverse isotropic linear elastic material.

Material symmetries of different tissues

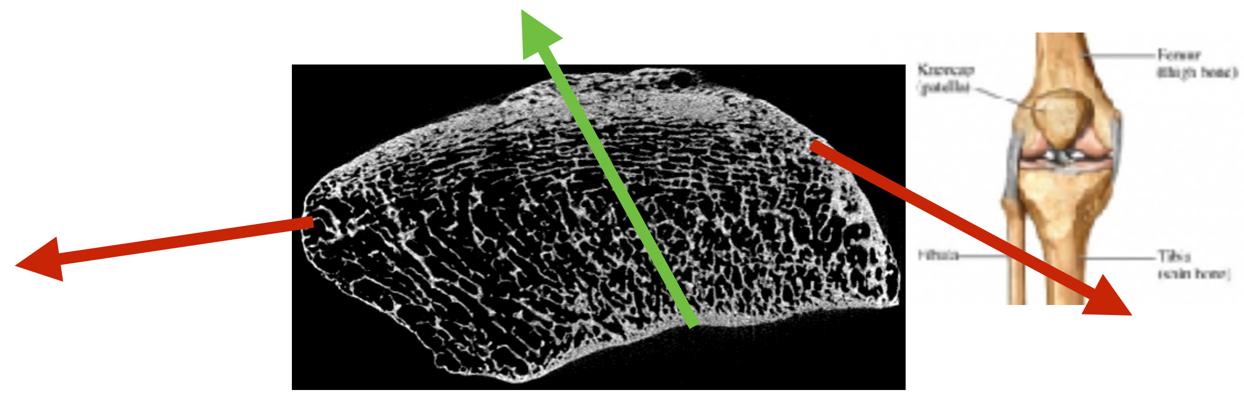
i) Isotropic -> cartilage ???

ii) Transverse isotropic -> ligament, tendon, bone?

iii) Orthotropic -> bone?

As previously mentioned, considering a material as isotropic is a convenient way to simplify the mechanical description as it will require less experiments to obtain the value of the parameters. Some tissues clearly do not show an isotropic mechanical behaviour so this approximation may induce an imprecise description leading to a false interpretation of the obtained mechanical description. The “art” of the (bio)mechanician is then to evaluate when certain approximations can be considered or not as acceptable.

Material symmetries of different tissues



sagittal view

tendon force

femora-patella contact force

One important aspect of biomechanics is then to characterise tissues through constitutive laws

$$\rho \frac{d\boldsymbol{v}}{dt} = div\ \boldsymbol{\sigma} + \rho \boldsymbol{b}$$

$$\sigma = \sigma(\varepsilon, \dot{\varepsilon}, \varepsilon_p, \ldots)$$

Elasticity -> σ = σ(ε) → Linear / **Non-linear**

There is often a confusion between non-linear elastic behaviour and large deformation. While in general, a material submitted to a large deformation will display a non-linear stress-strain relationship, we can find materials presenting this non-linear behaviour already at low strain or inversely some materials may present a linear elastic behaviour at high strain.

Biomechanics at the tissue level

i) Continuum mechanics (conservation laws)

ii) Constitutive laws (linear, non-linear)

iii) Tissue characterisation